Research directions for graduate students

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1 Representations of operator spaces

Let E be a Banach space. A general problem in Banach space theory is, *into which Banach spaces can V be embedded isometrically?* For example, if H is a Hilbert space, then every subspace of H is also a Hilbert space. So E can be embedded into H if and only if E is a Hilbert space with dimension less than or equal to the dimension of H (yawn).

Remarkably, for every Banach space E, there exists a Hilbert space H such that E can be embedded isometrically into $B(H)$ — the Banach algebra of bounded linear operators on H. This means that every Banach space is an "operator space".

Time for a definition: an ***operator space*** is a subspace $E \subseteq B(H)$ of the algebra of bounded operators on some Hilbert space H. Operator spaces inherit a linear structure and a norm from the ambient $B(H)$ and thus become normed spaces, and if they are closed subspaces then they are Banach spaces. But it is important to note that an operator space inherits additional structure from $B(H)$. Significantly, thinking of E as a subspace of $B(H)$ immediately gives a way with which to norm the spaces $M_n(E)$ of $n \times n$ matrices over E. Perhaps surprinsgly, these additional "matrix norms" contain information about the actual operators in E and how they act on H, not only about its structure as an abstract normed space.

It turns out that the same Banach space can be represented in different ways as an operator space, such that the corresponding operator spaces are isometrically isomorphic, but carry a different operator space structure. Thus, operator space theory can be conisdered to be a refinement or a generalization (depending on what you had for breakfast) of Banach space theory. This theory is highly developed, powerful and rich (see [9] and [10]).

Suppose that E is a Banach space and that $U_i : E \to E_i \subseteq B(H_i)$ ($i = 1, 2$) are two isometric linear isomorphisms. In what way are E_1 and E_2 related? Can we quantify the difference in their operator space structure? What conditions on E force the posibble operator space structures to be uniquely determined?

More questions: suppose that E is finite dimensional, can H be chosen to be finite dimensional, too? And if so, in what way does the geometric structure of E determine the minimal dimension of H such that E can be exhibited as a subspace of $B(H)$? (It is worth pointing out that we know that even if E is two dimensional, we may need an infinite dimensional H in order to represent E as a space of operators on H). Finally, under what circumstances does the operator space structure determine the action of the operators on the space up to unitary equivalence?

My colleagues and I studied questions of this flavour in a series of papers [2, 6, 11], but there are many exciting questions to explore.

2 The isomorphism problem for multiplier algebras

Let X be a set. A ***Hilbert function space*** on X is a Hilbert space H that is a linear subspace of the space of all functions $f : X \to \mathbb{C}$, such that point evaluation $x \mapsto h(x)$ $(h \in H)$ is a bounded linear functional for every $x \in X$. Note straight away, that the space $L^2(D)$ of Lebesgue square integrable functions on a domain $D \subseteq \mathbb{R}^d$ is *not* a Hilbert function space. A good example of a Hilbert function space is the space $L^2_a(D)$, which consists of all analytic functions on a bounded domain $D \subseteq \mathbb{C}$ which are also square integrable (it should be clear to you that $L^2_a(D)$ is a linear subspace of $L^2(D)$, but the facts that it is complete with respect to the L^2-norm and that point evaluation is bounded do require proofs).

Hilbert function spaces are not to be viewed merely as Hilbert spaces. The fact that point evaluation is a bounded functional connects between the Hilbert space structure and the function theoretic phenomena that occur in this space. This gives rise to a very rich interplay between operator theory and function theory. For a detailed introduction to the subject, see [1] (see also Chapter 6 in [16] for a crash preview).

Given a Hilbert function space H on a space X, we define its ***multiplier algebra*** to be the algebra of functions

$$\mathrm{Mult}(H) = \{f : X \to \mathbb{C} : fh \in H \text{ for all } h \in H\}.$$

A function f in the multiplier algebra gives rise to a ***multiplication operator*** $M_f : H \to H$ given by $M_f h = fh$. It is a pleasing consequence of the closed graph theorem that a multiplication operator is automatically bounded. The multiplier algebra $\mathrm{Mult}(H)$ can be equipped with the ***multiplier norm***

$$\|f\|_{\mathrm{Mult}} = \|M_f\|,$$

and becomes a Banach algebra, and in fact it is an operator algebra.

We are interested in Hilbert function spaces especially in the case where X is some kind of analytic variety and H consists of analytic functions. In this case $\mathrm{Mult}(H)$ also consists of analytic functions. It is intruiging to study the relationship between the structure of H as a Hilbert function space, the structure of $\mathrm{Mult}(H)$ as a Banach algebra (or as an operator algebra) and the geometric structure of X. Together with my colleagues and students I studied such problems in the papers [3, 4, 5, 7, 8]; see also the survey [12].

An important class of Hilbert function spaces are the ***complete Pick spaces***. These are Hilbert function spaces H_X that are constructed in a certain way on a subset $X \subseteq \mathbb{B}_d$ in the open unit ball in $\mathbb{C}^d$ (see any of the above cited papers or the survey [15] to see how this simple construction is carried out). One of our prototypical results says that if X and Y are analytic varieties in $\mathbb{B}_d$, then H_X and H_Y are isomorphic as Hilbert function spaces, if and only if $\mathrm{Mult}(H_X)$ and $\mathrm{Mult}(H_Y)$ are isometrically isomorphic (as Banach algebras), and that this happens if and only if X and Y are conformally equivalent, in the sense that there exists a conformal automorphism of the ball $\phi \in \mathrm{Aut}(\mathbb{B}_d)$ such that $\phi(X) = Y$. Thus, (in this setting) the Hilbert function space structure and the multiplier algebra structure determine one another completely, and both of them are complete invariants of the conformal geometry of the variety X. We also have results on the classifcation of the algebras $\mathrm{Mult}(H_X)$ up to *algebraic* isomorphism, which show that if we use mere algebra we encode a coarser kind of geometry on X. There are many open questions in this field, waiting to be explored.

I have also studied with my colleagues "noncommutative" versions of the above questions, in which we study similar questions but where we consider noncommutative analytic functions, which are basically like anlaytic functions but with the complex numbers replaced by complex matrices of arbitrary size; see [13, 14].

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