

A RESEARCH DIRECTION FOR STUDENTS: FAMILIES OF POLYNOMIALS

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The proposed research concerns basic properties of families of polynomials, such as their irreducibility and existence of roots. Given an irreducible (bivariate) polynomial $f(t, x)$ with rational coefficients, Hilbert's irreducibility theorem asserts that for infinitely many values $t_0 \in \mathbb{Q}$, the (univariate) polynomial $f(t_0, x)$ is irreducible. In the past century Hilbert's irreducibility theorem and its generalizations were of wide use, e.g. in studying statistics of polynomials, inverse Galois theory, and in many proofs (e.g. of Fermat's last theorem). It turns out that for most polynomials $f(t, x)$, much sharper bounds on the set of "bad" specializations $Red_f := \{t_0 \in \mathbb{Q} \mid f(t_0, x) \in \mathbb{Q}[x] \text{ is reducible}\}$ are known. In particular, it is known that in most cases either Red_f is finite(!) or it is the union of a finite set with the set $\{t_0 \in \mathbb{Q} \mid f(t_0, x) \text{ has a rational root}\}$.

For simplicity, we focus on polynomials of the form $f(t, x) = p(x) - t$ where $p(x)$ is a univariate polynomial with rational coefficients. Each such polynomial can be thought of as the family of polynomials obtained from p by varying the free coefficient. For such polynomials Red_f contains all values $t_0 = p(a)$ for $a \in \mathbb{Q}$, since then $f(t_0, x) = p(x) - p(a)$ is reducible and even has a rational root. We suggest the following problems:

- With an explicit list of exceptional polynomials p , show that the following holds for every nonexceptional p :

$$Red_{p(x)-t} = \bigcup_{i \in I} p_i(\mathbb{Q}) \cup \text{finite set},$$

where $P = \{p_i, i \in I\}$ is the set of first factors of p , that is, polynomials p_i over the rationals such that $p = p_i \circ p'_i$ for some polynomial p'_i .

- Determine the cardinality of the above set P . That is, how many first factors can p have? This relates to Ritt's theorems.
- Give a similar description for the set $Red_{p(x)-tx^i}$, i.e. the set of reducible polynomials when varying the coefficient of x^i .
- Apply this to estimating the cardinalities of the value sets $\#p(\mathbb{F}_q)$ for large primes q , where \mathbb{F}_q is the field with q elements.

We note that our current knowledge does provide answers for polynomials p whose composition factors are of large degree and the main difficulty is incorporating low degree composition factors. Analyzing low degrees might include programming, although this is not essential for the project.

The necessary background is a course in field theory. Knowledge of topics in algebraic number theory or commutative algebra (Dedekind domains) could be an advantage.

More research directions concerning families of polynomials will be given upon request.

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