# Research proposal

## 1   Very quick introduction

In broad terms, most of my research is centered around two subjects: i) the rigorous understanding of models arising in statistical and mathematical physics in the context of probability theory, see 1-3, and ii) the use of combinatorial and probabilistic techniques for the study of simplicial complexes as high-dimensional graphs, see 4-5. More specifically, some of the topics I have been working on include:

1. Geometry of general percolation models on the Euclidean lattice. Percolation is a classical model in statistical physics and probability theory which describes connected components of random graphs and is used to model many physical phenomena. The understanding of its geometry is therefore of great interest.

2. Random walks in random environments is a probabilistic model that describes the movement of particles on graphs, with transition probabilities that are also chosen randomly. Such models aim to describe the movement of particles in non-perfect medium, that is materials containing defects. The main goal is to analyze the asymptotic behavior of the random walk.

3. Random matrices, namely matrices whose entries are random variables, have been playing essential role in contemporary mathematics and science. Models of random matrices has found applications in numerous fields of mathematics, for example: probability theory, operator theory, number theory and combinatorics, as well as in most fields of science. The main goal is to understand the limiting distribution of eigenvalues and eigenvectors of different models of such matrices.

4. Growth models. Probabilistic growth models haven been used to model growth in a variety of real life phenomena such as population, tumours, bacteria, etc. The main goal is to understand the growth rate and limiting geometry of the aggregates.

5. Simplicial complexes are natural combinatorial and topological extensions of graphs and are thus extremely interesting objects. A flourishing area of research aiming at generalizing classical results and models from graphs to higher-dimensional complexes provides a diverse field of study. Some examples of possible directions include: (a) Study of probabilistic models on simplicial complexes such as percolation, random walks, Gaussian fields, etc; and (b) Analyzing the typical behavior of different models of random simplicial complexes.

6. High-dimensional expanders. Expanders are sparse graphs, i.e., with relatively few edges, which are highly connected. The construction of expanders have generated research in pure and applied mathematics and has many applications to computer science. In the last few years there has been growing interest in understanding the high-dimensional counterparts (in the sense of simplicial complexes) of expanders and in applications of such models to mathematics and computer science.

A bit more details on each of these subjects can be found in the following pages.

## 2 A bit more details

### 2.1 Percolation models

Percolation theory is a model in mathematics and statistical physics which describes the behavior of connected components in random graph, and is the simplest model in which the phenomena of phase transition can be observed.

For example, consider the 2-dimensional Euclidean lattice $\mathbb{Z}^2$ with edges between any pair of points $(x_1, y_1), (x_2, y_2)$ such that $|x_1 - x_2| + |y_1 - y_2| = 1$. Here is an illustration of part of the Euclidean lattice.
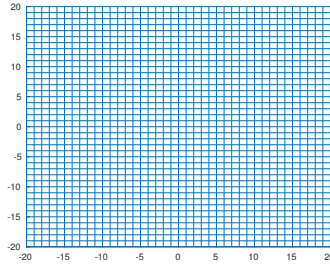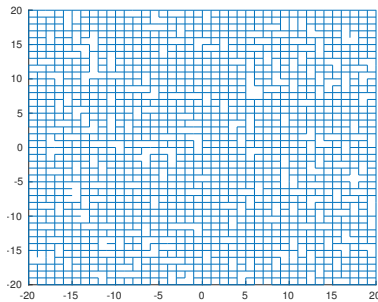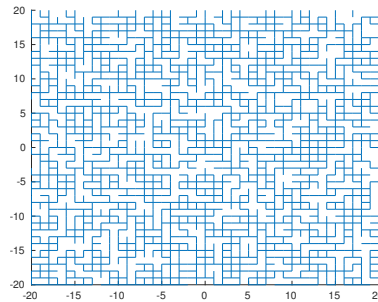


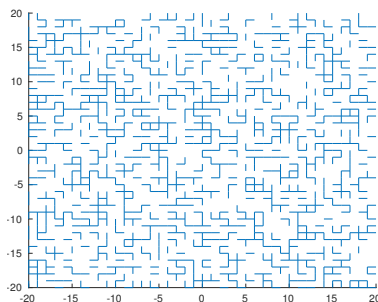Figure 1: Part of The Euclidean lattice $\mathbb{Z}^2$.

Let $p \in [0, 1]$. In Bernoulli percolation with parameter $p$ we delete each of the edges independently with probability $1 - p$, and keep it with probability $p$. Here are simulations for different values of $p$:
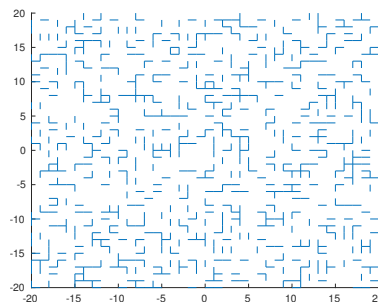


(a) Percolation with $p = 0.9$



(b) Percolation with $p = 0.7$



(c) Percolation with $p = 0.4$



(d) Percolation with $p = 0.25$

Observing the pictures, one can see that the graph structure in the first two examples ($p$ sufficiently large) is very different than the graph structure in the last two examples ($p$ sufficiently small). For example, one may check that in the first two cases there is an open path from left to right, while in the other two there is no such path. In fact, there is a special value $p_c$, called the critical value, in which the behavior changes drastically. The existence of different behavior for different values of $p$ is called (the existence of) phase transition.

I am mostly interested in understanding such phenomena in percolation models where there are dependencies between the state of different edges and in models where long range edges are present.

## 2.2 Random walks in random environments

The model of Random Walk in Random Environment (RWRE) is intended to capture the behavior of particle movement in a non-homogeneous medium (called the environment). Think for example on a particle moving inside a crystal. In a perfect crystal (under some assumption and looking only in its bulk) the crystal looks the same from each point and the forces acting on the particle are the same everywhere. Thus the motion can be approximated in a discretized model by the sum of i.i.d. random variables. However, crystals in nature have imperfections such as missing particles or excess of particles and therefore the motion is irregular.

In general it is very hard to understand the law that governs the motion of a particle in a non-homogeneous medium (the environment) and here is where the randomness of the media enters the picture. As it turns out if instead of talking about a specific environment, we discuss the case of a random one, we can learn much more on the law of the particle (under some assumption on the law of the environment).

The model of RWRE thus involves two components. The first is the environment which is randomly chosen, but kept fixed throughout the evolution of the random walk, and the second is the random walk which for a given environment is a time-homogeneous Markov chain with law that depends on the environment It is a crucial fact of the model that the environment is kept fixed throughout the evolution of the random walk.

For simplicity let us concentrate on the following simple example. Fix $0 < p_1, p_2 < 1$ and let $(\omega_x)_{x \in \mathbb{Z}}$ be i.i.d. random variables which are equal to $p_1$ with probability $0 < \alpha < 1$ and to $p_2$ with probability $1 - \alpha$. Given a fixed choice of $\omega = (\omega_n)_{n \in \mathbb{Z}}$, i.e. , the environment, we define a random walk $(X_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}$ which starts at $X_0 = 0$ with transition probabilities (see Figure 3 for an illustration).

$$P_\omega(X_{n+1} = x + 1 | X_n = x) = \omega_x, \qquad \forall x \in \mathbb{Z}, \, n \geq 0$$
$$P_\omega(X_{n+1} = x - 1 | X_n = x) = 1 - \omega_x, \qquad \forall x \in \mathbb{Z}, \, n \geq 0 \, .$$
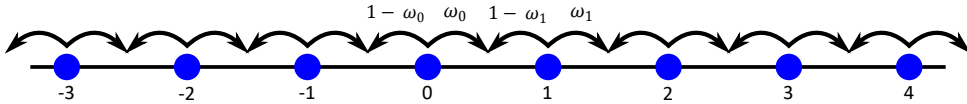


Figure 3: Illustration of an environment (transition probabilities) on part of $\mathbb{Z}$.

If $p_1 = p_2 = \frac{1}{2}$, then the environment is fixed and the random walk is just a simple random walks and therefore, by classical results, it almost surely return to 0 infinitely many times, and the typical distance from the origin at time $n$ is of order $\sqrt{n}$. If $p_1 = p_2 \in (1/2, 1)$, then the environment is fixed and the random walks has a drift to the right. In particular, it will only return to the starting point finitely many times almost surely and its distance from the starting point at time $n$ is of order $(2p - 1)n$. On the other hand, if $p_1 < \frac{1}{2} < p_2$, then the environment is random and the behavior of the random walk is very different. For example, for certain values of $\alpha$ the typical distance from the origin at time $n$ is $\log n$ instead of $\sqrt{n}$.

My research in RWRE is mainly focused on understanding the model in the high-dimensional case, i.e. on $\mathbb{Z}^d$ for sufficiently large $d$. In particular, I am interested in the effect of traps on the asymptotic behavior of the walk.

## 2.3 Random matrices

Matrices play an essential role in mathematics and science and it is therefore natural that probabilists have turned to study random variants of them, namely, matrices whose entries are random variables. In practice, the study of random matrices emerged from their applications. The first appearance of random matrices goes back to the work of Wishart 1928 who suggested a model for studying the covariance matrix of parameters associated with the population by replacing an empirical observation (from surveys) with random matrices. Then, starting in the 50's with the work of Wigner on statistical models for heavy-nuclei atoms, the subject has found many applications in physics: electromagnetic response of irregular metallic grains, transport properties in disordered quantum systems, quantum chaotic systems, QCD, random lattices and KPZ models, integrable systems, growth models, random Schrödinger operators and much more. There is also abundance of applications in mathematics: random operators, random graphs, free probability, zeroes of the Riemann zeta function, study of random permutations, etc.

The main objective of random matrix theory is to understand the typical behavior of the eigenvalues and eigenvectors of random matrices for many models of random matrices. Typical questions include:

1. What is the eigenvalue distribution in different scales?

2. What is the typical distance between eigenvalues?

3. What is the order of magnitude of the largest\smallest eigenvalue

4. Are the eigenvectors localized (most entries are very small and only few are large) or delocalized (all entries are roughly of the same order)?

The answers to this questions depends on several key properties of the random matrices under consideration: First, the field from which the entries are sampled; second, the ensemble of matrices under consideration (general, symmetric, Hermitian, unitary, etc); and third the joint distribution of the random variables. For example, if one discuss random Hermitian matrices (over the complex numbers) then all eigenvalues are real, while if one discuss general random matrices over the complex numbers, then typically all eigenvalues are complex.

As an example for a result in random matrix theory we state Wigner's semicircle theorem: Let $M_N$ be a real symmetric $N \times N$ random matrices whose upper triangular entries are i.i.d. with mean zero and variance 1, and denote by $\lambda_1, \lambda_2, \ldots, \lambda_N$ its eigenvalues. Then, the normalized eigenvalue statistics $\frac{1}{N} \sum_{i=1}^{N} \delta_{N^{-1/2}\lambda_i}$ converges to the semicircle law. See Figure 4 for an illustration.
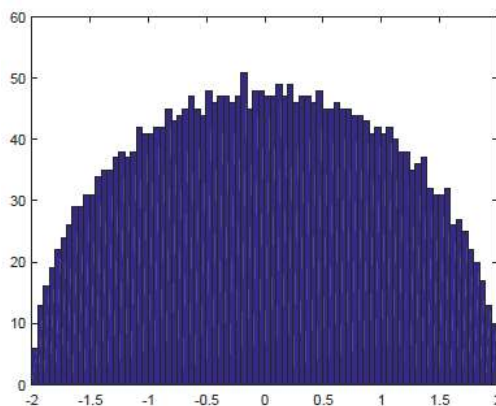


Figure 4: Histogram of the normalized eigenvalue statistics of a real symmetric matrix whose upper triangular entries are i.i.d. random variables such that $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$.

## 2.4 Growth models

Growth models aim to describe different growth phenomena such as the growth of bacteria, plants, coral, snowflake, crystals, tumours, etc. In the probabilistic setting one usually add in each step a new particle to the currently existing aggregate in a random way. For simplicity, we concentrate on discrete models, both in time and in space, on the two-dimensional Euclidean lattice $\mathbb{Z}^2$.

A typical random growth model on $\mathbb{Z}^2$, is given by a sequence of random subsets $(A_n)_{n \geq 0}$ of $\mathbb{Z}^2$ such that $A_n \subset A_{n+1}$, where $A_{n+1}$ is obtained from $A_n$ by adding a random particle (vertex in $\mathbb{Z}^2$) from the boundary of $A_n$

$$\partial A_n = \{v \in \mathbb{Z}^2 \ : \ v \notin A_n \text{ and } \exists w \in A_n \text{ which is a neighbor of } v\}.$$

For example, if at some point the set is given by
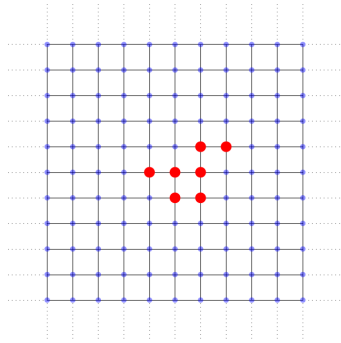


then, the next set is obtained by adding one of the boundary points (colored in green in the following picture) to the existing set.
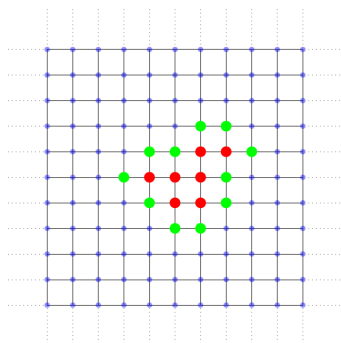


Different choice of randomness leads to different limiting behavior, see Figure 5, and the main goal here is to analyze the growth rate and the limiting shape of the sets $(A_n)_{n \geq 0}$.
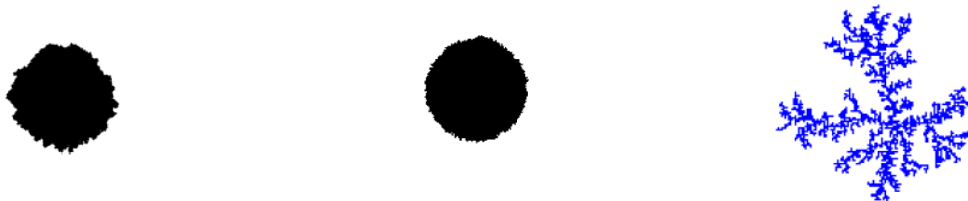


Figure 5: Simulations of different growth models. On the left the Eden model. In the middle Internal Diffusion Limited Aggregation (IDLA). On the right Diffusion Limited Aggregation (DLA).

## 2.5   Simplicial complexes

Simplicial complexes are natural extensions of graphs. Given a finite set $V$, a simplicial complex $X$ on the vertex set $V$ is a family of subsets of $V$ which is closed under inclusion. That is, if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in \tau$. A simplicial complex is composed of subsets of size 1 (called vertices), subsets of size two (called edges), subsets of size three (called triangles) and so on. In general a set of size $j + 1$ is called a $j$-dimensional cell (or $j$-cell for short). For an example of a simplicial complex, see Figure 6.



(a) An illustration of a simplicial complex $X$ with vertices, edges and triangles.

(b) The vertex set of $X$ is $\{0, 1, 2, \ldots, 9\}$.

(c) The edges of $X$ appear in blue.
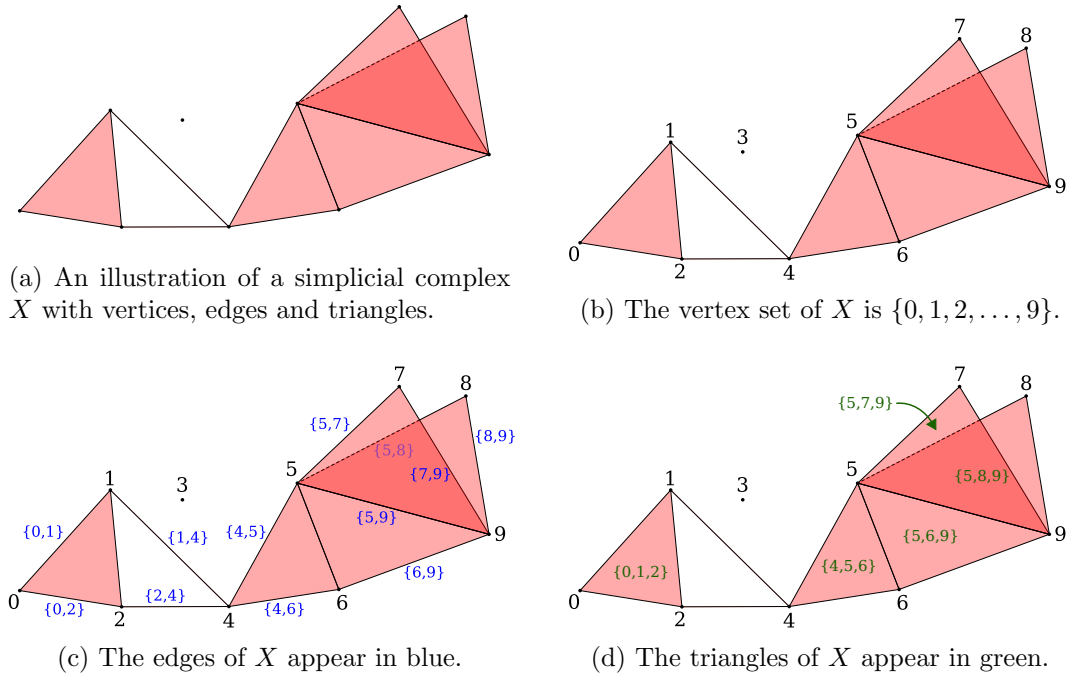
(d) The triangles of $X$ appear in green.

Figure 6: Illustration of a simplicial complex.

Note that a simplicial complex that contains only vertices and edges is a graph, and it is precisely in this sense that simplicial complexes generalize graphs.

Since graphs play a crucial role in many fields of mathematics and science there are numerous results addressing their behavior and properties. Consequently, there are various of directions in which one can try to generalize the results from graph theory to the realm of simplicial complexes. Some examples include:

1. Generalizing models of random graphs to random simplicial complexes and study their behavior.

2. Investigate high-dimensional versions of random walks on simplicial complexes.

3. Study high-dimensional variants of expander graphs.

4. Study random simplicial complexes as topological objects.

5. Research applications of simplicial complexes to computer science.

6. Find applications of simplicial complexes to statistical physics.