Isoperimetric Inequalities, Concentration of Measure, Convexity, and their applications

(Winter Semester, Course 106433, “Topics in Functional Analysis”)

Instructor: Emanuel Milman.
Time: Wednesdays, 14:30 - 17:30. Place: TBD, we will start via Zoom.
Language: pending request, the course will be given in English.

This course is not directly related to Functional Analysis in the traditional sense, but more to Geometry and Analysis. From this respect the course will be inter-disciplinary.

Given a metric-measure space, there are numerous ways to study the interaction between metric and measure. In this course, we will learn about these various manifestations, putting an emphasis on applications and their relation to other fields such as Brunn-Minkowski Theory (the study of convex bodies in Euclidean space), the spectrum of the Laplacian, heat semi-groups, (generalized) Ricci curvature, and more.

Our starting point will be that of isoperimetric inequalities, which provide a lower bound on the surface area of a set having a prescribed measure (the Euclidean isoperimetric inequality, stating that among all polygons in the plane of a given area, the circle has minimal perimeter, was known already to the Ancient Greeks). We will prove the isoperimetric inequalities in the classical spaces (Euclidean, on the Sphere) and for the Gaussian measure, and will relate them to the Brunn-Minkowski inequality and its generalizations. We will prove the Borell / Sudakov—Tsirelson theorem on Gauss space, and might touch upon the Gromov-Levy theorem on Riemannian manifolds with positive Ricci curvature. We will see how symmetrization produces isoperimetric and other inequalities.

The smaller siblings of Isoperimetric Inequalities are Concentration Inequalities, which lead to the Concentartion of Measure Phenomena in high-dimension. In between these two extremes lie analytic inequalities such as the Poincare and log-Sobolev inequalities. We will learn about how all of these inequalities are related (e.g. Cheeger’s inequality via the co-area formula). We will show that positive (Ricci) curvature is responsible for creating a spectral-gap (Lichnerowicz and Brascamp-Lieb inequalities).

The aim will be to learn and have fun, and I will try to navigate the course with maximum flexibility. The grade will be based on participation and a home exam based on the exercises given throughout the course.

Prerequisites: nothing in particular, basically Mathematical maturity. I will assume basic familiarity with the Lebesgue measure, notions in Probability, and Calculus proficiency. Basic familiarity with Riemannian geometry, Banach spaces and heat-flow might be helpful, but is definitely not mandatory. The emphasis will be on ideas and **not** on foundations nor on overemphasis of rigourosity.

Tentative Syllabus:

Brunn-Minkowski theory of convex bodies and its applications
The isoperimetric inequality in Euclidean space, the sphere and Gauss space, and concentration inequalities.
The Johnson-Lindenstrauss lemma on dimension-reduction.
The co-area formula and the relation between isoperimetry and analytic inequalities such as Poincare, Sobolev and log-Sobolev inequalities.
The spectrum of the Laplacian, spectral-gap and the Faber-Krahn inequality.
The heat equation on a weighted Riemannian manifold and its application for proving functional inequalities.
Generalized Ricci curvature and it relation to the heat equation and concentration of measure. The Lichnerowicz and Brascamp-Lieb inequalities.
Time permitting, the theory of Optimal-Transport and applications for proving isoperimetric, functional and concentration inequalities.